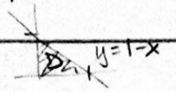
# Lessons 23 & 24 Review

1. $\iint_S 6xy\,dS$, $S$ = portion of plane $x+y+z=1$ in 1st octant

$z = 1-x-y$




$\vec{r}(x,y) = \langle x, y, 1-x-y \rangle$

$$\int_0^1 \int_0^{1-x} 6xy\sqrt{1+1+1}\;dy\,dx = \sqrt{3}\int_0^1 3x(1-x)^2\,dx$$
$$= \sqrt{3}\int_0^1 3x(x^2-2x+1)\,dx$$
$$= \sqrt{3}\int_0^1 3x^3-6x^2+3x\,dx$$
$$= \sqrt{3}\left(\tfrac{3}{4}-2+\tfrac{3}{2}\right) = \boxed{\tfrac{\sqrt{3}}{4}}$$

2. $\vec{F} = \langle x, y, z \rangle$, $S = x^2+y^2+z^2=4$, positive orientation

$\vec{r}(\phi,\theta) = \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle$

$\vec{r}_\phi = \langle 2\cos\phi\cos\theta, 2\cos\phi\sin\theta, -2\sin\phi \rangle$

$\vec{r}_\theta = \langle -2\sin\phi\sin\theta, 2\sin\phi\cos\theta, 0 \rangle$

$\vec{r}_\phi \times \vec{r}_\theta = \langle 4\sin^2\phi\cos\theta, 4\sin^2\phi\sin\theta, 4\sin\phi\cos\phi \rangle$
$= 4\sin\phi\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle$

$$\int_0^{2\pi}\int_0^{\pi} \langle 2\sin\phi\cos\theta, 2\sin\phi\sin\theta, 2\cos\phi \rangle \cdot 4\sin\phi\langle \sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi \rangle\,d\phi\,d\theta$$
$$= \int_0^{2\pi}\int_0^{\pi} 8\sin\phi\,d\phi\,d\theta = 16\pi\,(-\cos\phi)\Big|_0^{\pi} = \boxed{32\pi}$$

3. $\iint_S z\,dS$, $S$ = upper hemisphere of sphere with radius 3

$z = \sqrt{9-x^2-y^2}$

$\frac{\partial z}{\partial x} = \frac{-x}{\sqrt{9-x^2-y^2}}$, $\frac{\partial z}{\partial y} = \frac{-y}{\sqrt{9-x^2-y^2}}$

$\sqrt{1+\left(\frac{\partial z}{\partial x}\right)^2+\left(\frac{\partial z}{\partial y}\right)^2} = \frac{3}{\sqrt{9-x^2-y^2}}$

$$\int_0^{2\pi}\int_0^3 3\,r\,dr\,d\theta = 2\pi\left(\tfrac{3}{2}(3)^2\right) = \boxed{27\pi}$$

4. $\vec{F} = \langle z, y, x \rangle$, $S$: $x = u\cos v$, $y = u\sin v$, $z = v$, $0 \le u \le 1$, $0 \le v \le \pi$, downward orient.

$\vec{r}_u = \langle \cos v, \sin v, 0 \rangle$

$\vec{r}_v = \langle -u\sin v, u\cos v, 1 \rangle$

$\vec{r}_u \times \vec{r}_v = \langle \sin v, -\cos v, u \rangle \leftarrow$ positive

$\vec{n} = \langle -\sin v, \cos v, -u \rangle \leftarrow$ negative

$$\int_0^{\pi}\int_0^1 -v\sin v + u\sin v\cos v - u^2\cos v\;du\,dv$$
$$= \int_0^{\pi} -v\sin v + \tfrac{1}{2}\sin v\cos v - \tfrac{1}{3}\cos v\,dv$$

$w = -v$ $\quad dx = \sin v\,dv$

$dw = -dv$ $\quad x = -\cos v$

$$= \left[v\cos v - \cancel{\sin v}\right]_0^{\pi} + \left[\tfrac{1}{4}\sin^2 v\right]_0^{\pi} - \left[\tfrac{1}{3}\sin v\right]_0^{\pi} = \boxed{-\pi}$$

5. $\vec{F} = \langle \underset{P}{xy}, \underset{Q}{2yz}, \underset{R}{xyz} \rangle$, $S: z = x^2+y^2$ above $z=0$ below $z=4$, upward orientation

$f_x = 2x, \quad f_y = 2y$

$$\iint_D -2x^2y - 4y^2(x^2+y^2) + xy(x^2+y^2)\,dA = \int_0^{2\pi}\int_0^2 \left(-2r^3\cos^2\theta\sin\theta - 4r^4\sin^2\theta + r^4\sin\theta\cos\theta\right) r\,dr\,d\theta$$

$$= \int_0^{2\pi} -\frac{2}{5}(2)^5\cos^2\theta\sin\theta - \frac{4}{6}(2)^6\sin^2\theta + \frac{1}{6}(2)^6\sin\theta\cos\theta\,d\theta$$

$$= \int_0^{2\pi} -\frac{32}{5}\cos^2\theta\sin\theta - \frac{128}{3}\sin^2\theta + \frac{32}{3}\sin\theta\cos\theta\,d\theta$$

$$= +\frac{32}{5}\left(\frac{1}{3}\cos^3\theta\right) - \frac{64}{3}\left(\theta - \frac{1}{2}\sin 2\theta\right) + \frac{32}{3}\left(\frac{1}{2}\sin^2\theta\right)\Big|_0^{2\pi}$$

$$= \boxed{-\frac{128\pi}{3}}$$

6. $\iint_S y+z\,dS$, $S: z = 4-y$ above $x^2+y^2 = 3$

$$\iint_D [y + (4-y)]\sqrt{1+1+0}\,dA = 4\sqrt{2}\iint_D A = 4\sqrt{2}\,(3\pi) = \boxed{12\sqrt{2}\,\pi}$$